Dissipative systems

This chapter on dissipative systems is to a great extent based on reference [Oh].

A dissipative system is described by ordinary differential eqations where the flow in phase space contracts. In classical mechanics this is the case for a system with friction, where energy goes into the exterior system.

$$\begin{aligned} \dot{x}_1 &= f_1(t, x_1, x_2, \dots, x_n) \\ \dot{x}_2 &= f_2(t, x_1, x_2, \dots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(t, x_1, x_2, \dots, x_n) \end{aligned} \qquad n\text{-dimensional system}$$

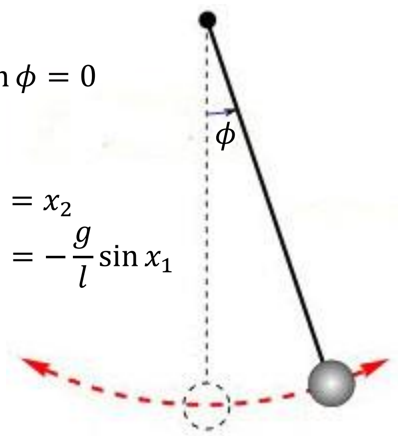
If all f_k are independent of time (t) the system is autonomous, $\boldsymbol{x}(t_0) = (x_1, x_2, \dots, x_n)$ determines a unique future (assuming that f_k fullfills Lipschitz conditions) which will be assumed henceforth.

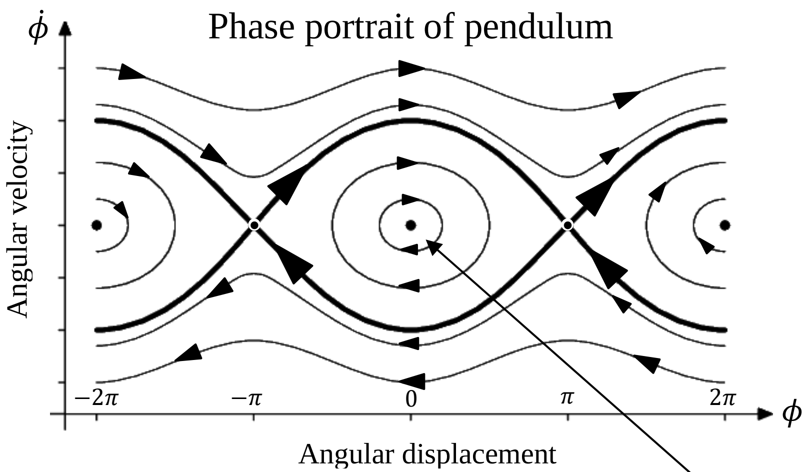
These systems are called **flows**. If both history and future is determined by $\boldsymbol{x}(t_0)$, it's called **invertible**.
A plot of x_k against t is called a **time series**. Dynamics is best analyzed using phase space representations where states are represented by ponts in $\mathbb{R}^n$. **Trajectories** and **orbits** are solutions $\boldsymbol{x}(t)$ plotted for $t > 0$. Flows are trajectories with directions given by increasing t. Trajectories do not cross if f_k independent of t.

Non-autonomous systems can be handled by introducing t as a state variable on an extra axis, this makes the system autonomous with no crossings. A plot of all possible trajectories is a **phase portrait**.

$$\ddot{\phi} + \frac{g}{l}\sin\phi = 0$$

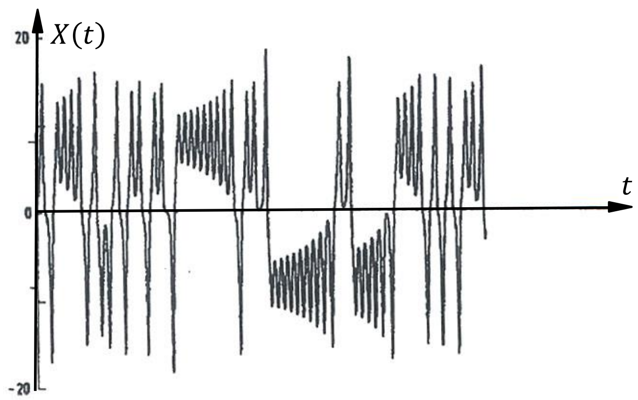
$$\begin{cases} x_1 = \phi \\ x_2 = \dot{\phi} \end{cases} \rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{g}{l}\sin x_1 \end{cases}$$





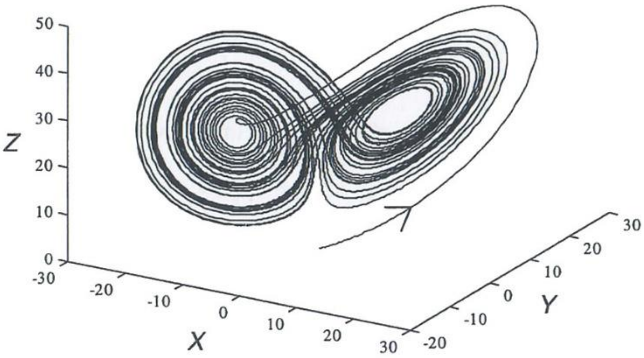
In the Lorenz system (X, Y, Z) all trajectories are converging towards the attractor as $t \to \infty$. The attractor becomes like a phase portrait on its own

Elliptic fixpoint is marginally stable. Orbits don't grow or shrink towards it.



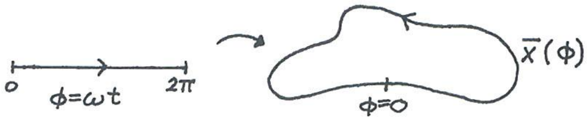
$X(T)$ jumping between two values at irregular intervals

Strange attractor of the Lorenz system

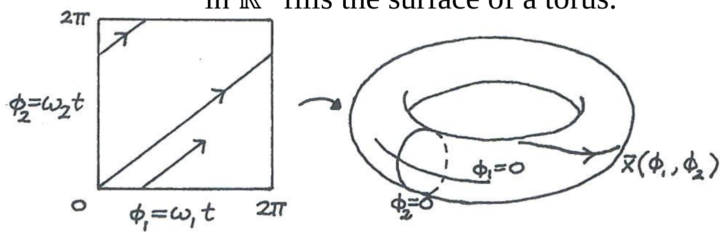


Trajectories of dynamical systems $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$

- When the system is in **equilibrium** at a **fixed point** $\boldsymbol{x}^*$, there is a stedy state solution
  $\forall t > t_0 : \ \boldsymbol{x}(t) = \boldsymbol{x}(t_0) = \boldsymbol{x}^* \quad \rightarrow \quad \boldsymbol{f}(\boldsymbol{x}^*) = 0.$
  The pendulum has a fixed point when it hangs down $(\phi, \dot{\phi}) = (0,0) + k \cdot (2\pi, 0)$ and
  an fixed point when it hangs upside down, pointing upwards $(\phi, \dot{\phi}) = (\pi, 0) + k \cdot (2\pi, 0)$.
- **Periodic orbits** or **cycles,** for flows means loops in phase space.
  A periodic orbit has a **period**, the smallest T such that $\forall t : \ \phi(t+T) = \phi(t)$
  which can be parametrized with an angle α and an angular velocity $x(t) = x(\alpha(t)) \ , \ \alpha = \omega t$
- **Quasi-periodic** orbits can be parametrized with two or more angles:
  $x(t) = x(\alpha_1(t), \alpha_2(t)) \ , \ \begin{cases} \alpha_1 = \omega_1 t \\ \alpha_2 = \omega_2 t \end{cases}$ where $\dfrac{\omega_1}{\omega_2} \in \mathbb{R} \setminus \mathbb{Q}$ (irrational)
  ($\omega_1/\omega_2 \in \mathbb{Q}$ would result in a periodic orbit)



A 2-frequency quasi-periodic orbit in $\mathbb{R}^3$ fills the surface of a torus.

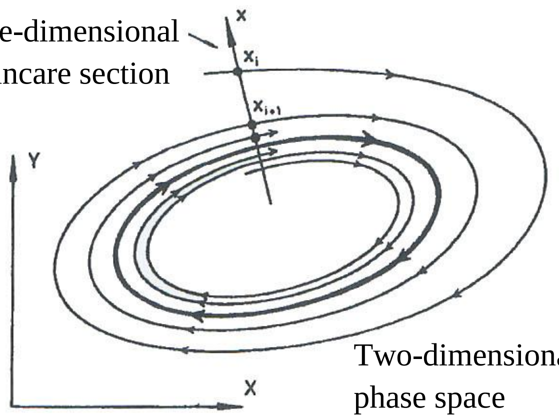


- Then there are trajectories that approach a fixed point, cycle or quasi-periodic orbit asymptotically.
- And finally there are chaotic trajecories with SIC.

Poincaré maps

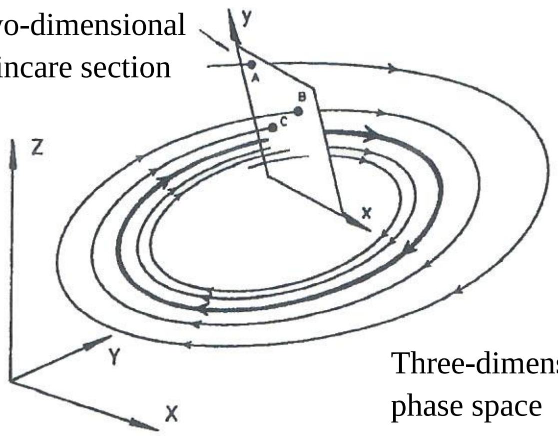
Dynamics of complicated flows can be simplified in a way that retains essential dynamic properties by using Poincaré maps that look at intersections between an $(n-1)$-dimensional surface $h(x_1, x_2, \dots, x_n) = 0$ and the trajectories of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$. Only intersections in one direction through the surface are considered.

One-dimensional Poincare section



Two-dimensional phase space

Two-dimensional Poincare section



Three-dimensional phase space

The dimension is reduced by one and the dynamics is simplified to a discrete map with similar dynamics.

Stability analysis

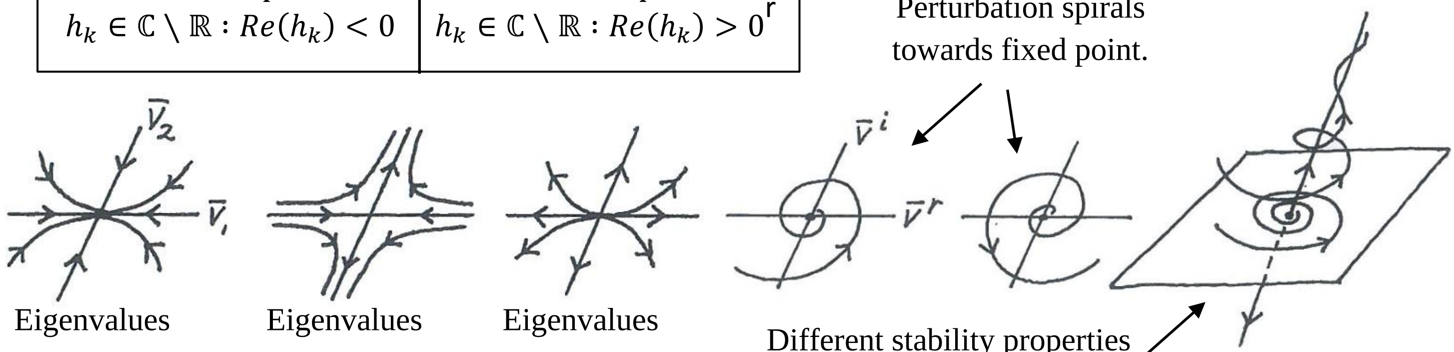
What was said about stability for discrete maps can be used for continuous maps as well. Start with a trajectory $\boldsymbol{x}(t)$ and a small perturbation at t_0, $\boldsymbol{\delta x}(t_0)$. This gives rise to another trajectory $\tilde{\boldsymbol{x}}(t) = \boldsymbol{x}(t) + \boldsymbol{\delta x}(t)$. Exponetial separation $|\boldsymbol{\delta x}(t)| / |\boldsymbol{\delta x}(t_0)| \propto e^{\lambda t}$ is the sign of chaos. Exponential separation is only for a limited time when the trajectories are bounded but extended local exponential separation can still go on by taking place on a strectched and folded manifold.

Let a box surround the initial value $x_k(0) \pm \Delta x_k(0)$ with volume $\delta V(0)$ and follow all trajectories in the ensemble, $\tilde{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{\delta x} \quad \rightarrow \quad \dot{\boldsymbol{x}} + \dot{\boldsymbol{\delta x}} = \boldsymbol{f}(\boldsymbol{x} + \boldsymbol{\delta x})$. Taylor expand for small perturbations → $\dot{\boldsymbol{\delta x}} = \boldsymbol{Df}(\boldsymbol{x})\boldsymbol{\delta x}$ with Jacobian matrix $\boldsymbol{Df}(\boldsymbol{x})$ which varies with the point of evaluation.
For eigenvector $\boldsymbol{v}_k$ with eigenvalue h of $\boldsymbol{Df}$: $(\boldsymbol{Df})\boldsymbol{v}_k = h\boldsymbol{v}_k$ where h is a solution to $\det(\boldsymbol{Df} - h\boldsymbol{I}) = 0$, for $\boldsymbol{\delta x}(0) = \varepsilon \boldsymbol{v}$: $\boldsymbol{\delta x}(\Delta t) = \boldsymbol{\delta x}(0) \cdot e^{h\Delta t} = \varepsilon e^{h\Delta t}\boldsymbol{v}.$ For a general perturbation $\boldsymbol{\delta x}(\Delta t) = \sum_{k=1}^{n} \varepsilon_k e^{h_k \Delta t}\boldsymbol{v}_k$.

Stability is decided by the eigenvalue with largest real part h_1:

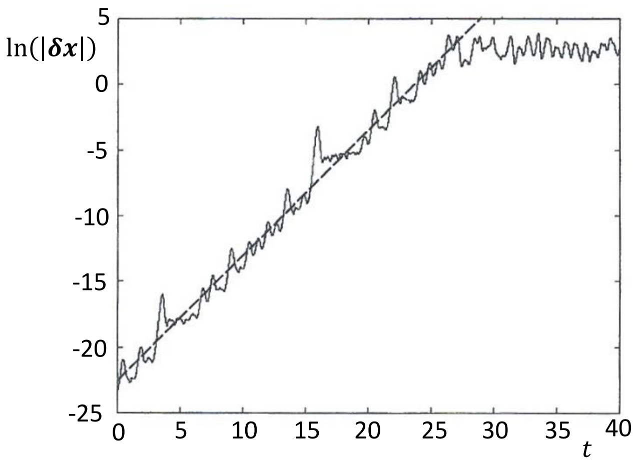
Stable	Unstable	Undecided
$Re(h_1) < 0$	$Re(h_1) > 0$	$Re(h_1) = 0$

Inward spiral	Outward spiral
$h_k \in \mathbb{C} \setminus \mathbb{R} : Re(h_k) < 0$	$h_k \in \mathbb{C} \setminus \mathbb{R} : Re(h_k) > 0$ r



Lyapunov exponents

The exponential separation of nearby trajectories is given by the Lyapunov exponent λ. Nearby points in phase space separates as $|\boldsymbol{\delta x}(t_2)|/|\boldsymbol{\delta x}(t_1)| \approx e^{\lambda(t_2 - t_1)} \quad \rightarrow \quad \ln(|\boldsymbol{\delta x}(t)|) \approx \lambda t + \text{const.}$



The diagram shows $\ln|\boldsymbol{\delta x}(t)|$ relative to another orbit in the Lorenz system.

The mean slope gives the largest Lyapunov exponent of the system.

When distances exceed the attractor size the exponential separation breaks down.

The Lyapunov exponents turn out to be the identical for most initial conditions lying in an attractor’s basin of attraction

Stretching and folding was the basic mechanism for chaos in 1-dimensinal iterations and so it is for higher-dimensional continuous systems as well. Stretching gives the exponential separation while folding is a consequence of non-linearity that stabilizes the system to a limited volume. In linear systems there is only stretching and no chaos.

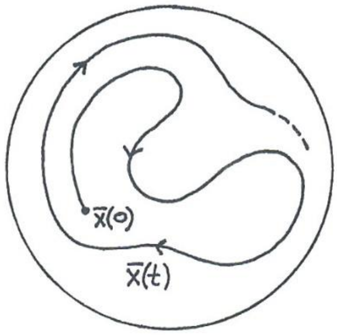
For dissipative systems there is stretching in one direction and compression in others so that volume decreases while trajectories diverge.

The criterions for a chaotic trajectory are:

- It is bounded
- Neither a fixed point, periodic, or quasi-periodic and it does not approach such a trajectory.
- It has a positive Lyapunov exponent.

Chaos and the dimension of phase space

The non-crossing of orbits forbids chaos in 1-dimensional systems.



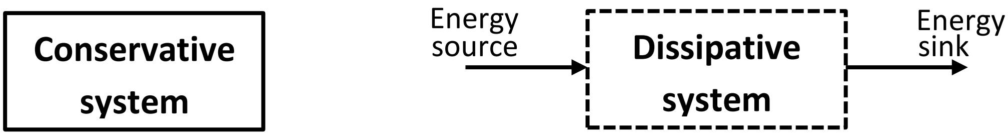
> Try to continue the trajectory so that it never closes and never leaves the circle and you will find it trapped in narrow tunnels, leaving less and less room for exponential separation.
>
> There can be no chaos in 2-dimensional flows

A mathematical proof that no chaotic trajectories can exist in 2-dimensional phase space is given by **Poincare-Bendixsons theorem**, an important theorem in non-linear dynamics.

For 3-dimensional flows and in higher dimensions chaos is always a possibility since trajectories can pass each other by going below or above in some extra dimension.

Definition of dissipative systems

A physical system where energy flows in and out of the system is called **dissipative**.



Friction is an energy sink that takes energy out of a system and transforms it to heat in the surroundings. Another example of a dissipative system is the body. We eat food as an energy source and we radiate heat.

Dissipative systems tend to have a behavior where they end up in their normal ways after a disturbance. This normal behavior need not be a fixed state or a periodic state, it can be chaotic but chaotic in a well-defined manner given by an attractor to the system. Ensembles of trajectories in phase space often shrinks toward an attractor with non-integer dimension, Cantor-like structure and self-similarity at different scales. Volume contraction in phase space gives a strict mathematical definition of dissipative systems.

A dissipative flow is given by a set of autonomous differential equations $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ for which a volume element enclosed by a surface S in phase space is not an invariant, usually it contracts:

$$\frac{dV}{dt} = \int_V \left(\sum_{k=1}^{n} \frac{\partial \boldsymbol{f}_k}{\partial \boldsymbol{x}_k} \right) d^n\boldsymbol{x} < 0$$

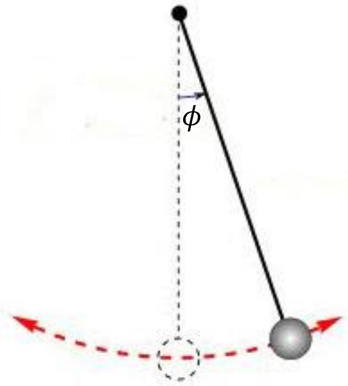
For the Lorenz system:

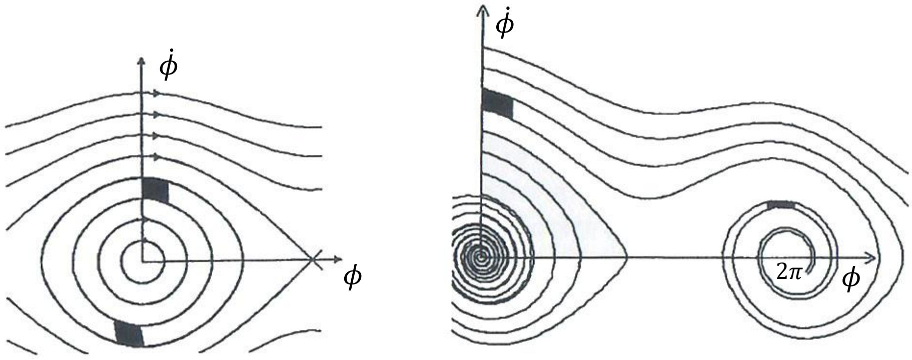
$$\begin{cases} \dot{X} = -\sigma X + \sigma Y \\ \dot{Y} = -XZ + rX - Y \\ \dot{Z} = XY - bZ \end{cases} \quad \rightarrow \quad \frac{dV}{dt} = -(\sigma + 1 + b)V < 0 \quad (\sigma > 0, b > 0)$$

As volume decreases, trajectories will approach a limit region. The dynamics before the attractor is approached is called **transient**, a temporary phase that fades away and which is often ignored.

Pendulum

A pendulkum withoput friction is conservative $\ddot{\phi} + (g/l) \cdot \sin\phi = 0$, energy flows between gravitational potential energy and kinetic energy. All energy flows are within the system. With friction such as air resistance there will be a damping term $\gamma\dot{\phi}$ that opposes the motion $\ddot{\phi} + \gamma\dot{\phi} + (g/l) \cdot \sin\phi = 0$. Energy is lost to heat in the surrounding air, it's a dissipative system.



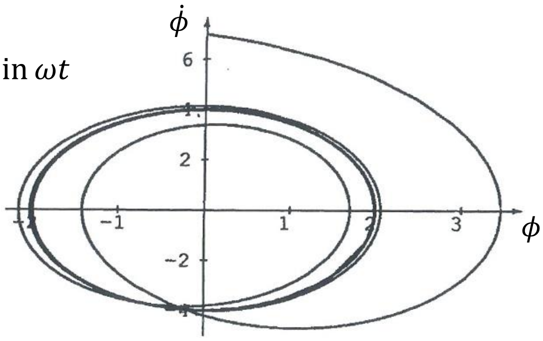


Phase portraits of undamped and damped pendulum.

Systems with an energy sink and no energy source like the damped pendulum will come to rest at a fixed point attractor. To get more interesting motion you need an energy source. This could be a periodic driving force accomplished by charging the pendulum and putting it in an oscillating electric field with frequency ω.

$$\ddot{\phi} + \gamma\dot{\phi} + (g/l)\sin\phi = A\sin\omega t$$

Equation of a pendulum with damping and a driving force.



Trajectory in phase space with an elliptic periodic attractor.

Note that the system is non-autonomous (time dependent) and that orbits in the $(\phi, \dot{\phi})$-space cross themselves.

With time a state variable $(x_1, x_2, x_3) = (\phi, \dot{\phi}, t)$ the system will be autonomous without crossing trajectories. The orbits of the figure will be extended into an extra perpendicular dimension that represents time and the orbits will no longer cross each other.

Linearizing $\sin\phi \approx \phi$ for small angles gives an equation that can be solved with elementary functions.

$$\phi(t) = \phi_p(t) + \phi_h(t) = C_1 \sin\omega t + C_2 e^{-\lambda t/2} \sin(\omega_0 t + \phi_0)$$

The asymptotic trajectory is given by the periodic solution $\phi_p(t) \propto \sin\omega t$.

Every orbit of the linear pendulum with restoring force $\propto \phi$ shrinks to the periodic orbit, a periodic attractor. More about the dynamics of a driven damped pendulum can be found on this link.

Volume contraction

A system is conservative if there are coordinates in phase space for which no ensemble of points change volume as time goes. If there can be such a volume change it's dissipative.

Look at a small volume $\Delta V(t)$, a box around a point $\boldsymbol{x}(t)$ with sides ε_k along eigenvector k of the Jacobian. At time Δt the sides will be $\varepsilon_k e^{h_k \Delta t}$ where $h_k(t)$ is eigenvalue k.

$$\frac{\Delta V(t+\Delta t)}{\Delta v(t)} = \prod_{k=1}^{n} \frac{\varepsilon_k e^{h_k \Delta t}}{\varepsilon_k} = \exp\left(\sum_{k=1}^{n} h_k \cdot \Delta t\right)$$

$$\frac{\ln(\Delta V(t+\Delta t)) - \ln(\Delta V(t))}{\Delta t} = \sum_{k=1}^{n} h_k$$

$$\frac{d}{dt}\ln \Delta V = \sum_{k=1}^{n} h_k$$

$$\frac{1}{\Delta V}\frac{d}{dt}\Delta V = \sum_{k=1}^{n} h_k(t) \quad \text{(Volume contraction rate changes along the trajectory)}$$

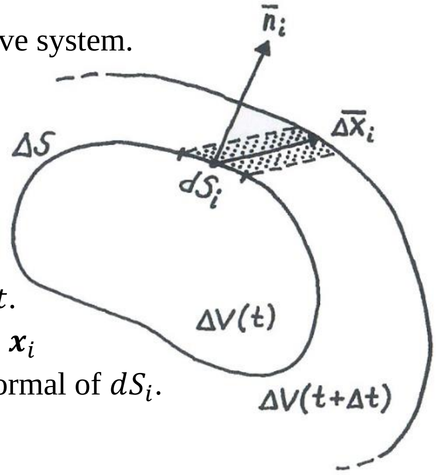
$\sum h_k < 0 \quad \rightarrow$ Phase space volume contracts locally around $\boldsymbol{x}$
$\sum h_k > 0 \quad \rightarrow$ Phase space volume expands locally around $\boldsymbol{x}$
This is true even if $h_k \in \mathbb{C}$ since h_k come in conjugate pairs with real sum.

Application to the pendulum

$$\ddot{\phi} + \gamma\dot{\phi} + \frac{g}{l}\sin\phi = 0 \quad \begin{cases} x_1 = \phi \\ x_2 = \dot{\phi} \end{cases} \rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{l}\sin x_1 - \gamma x_2 \end{cases} \rightarrow \boldsymbol{Df}(\boldsymbol{x}) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\gamma \end{pmatrix}$$

Eigenvalues: $h_{1,2} = -\frac{\gamma}{2} \pm \sqrt{(\gamma/2)^2 - \frac{g}{l}\cos x_1} \quad \rightarrow \quad \frac{1}{\Delta V}\frac{d}{dt}\Delta V = -\gamma$

Without damping and no driving force, no volume contraction and a conservative system.
With damping $\gamma > 0$ volume contracts to zero and a dissipative system.



Alternative method without decomposition into eigenvalues

There is a simpler method that don't need decomposition and eigenvalues.

The figure shows a phase space volume ΔV evolving from time t to time $t + \Delta t$.
The surface ΔS of ΔV is divided into sections dS_i. A point on dS_i evolves from $\boldsymbol{x}_i$ to $\boldsymbol{x}_i + \Delta\boldsymbol{x}_i$, resulting in a volume change $\Delta\boldsymbol{x}_i \cdot \boldsymbol{n}_i\, dS_i$ with $\boldsymbol{n}_i$ being the unit normal of dS_i.

Integration gives a volume change:

$$\Delta V(t+\Delta t) - \Delta V(t) = \int_{\Delta S} \Delta\boldsymbol{x} \cdot \boldsymbol{n}\, dS \quad \rightarrow \quad \text{(divide by } \Delta t\text{)}$$

$$\frac{d}{dt}\Delta V = \int_{\Delta S} \dot{\boldsymbol{x}} \cdot \boldsymbol{n}\, dS \quad \rightarrow \quad \text{(Gauss' theorem)}$$

$$\frac{d}{dt}\Delta V = \int_{\Delta V} \mathrm{div}(\dot{\boldsymbol{x}})\, dV$$

$\mathrm{div}(\dot{\boldsymbol{x}}) \equiv \sum_{k=1}^{n} \frac{\partial \dot{x}_k}{\partial x_k}$ is the **Lie derivative**

If ΔV is small and $\mathrm{div}(\dot{\boldsymbol{x}})$ can be assumed constant in ΔV we get: $\frac{1}{\Delta V}\frac{d}{dt}\Delta V = \sum_{k=1}^{n} \frac{\partial \dot{x}_k}{\partial x_k}(t)$

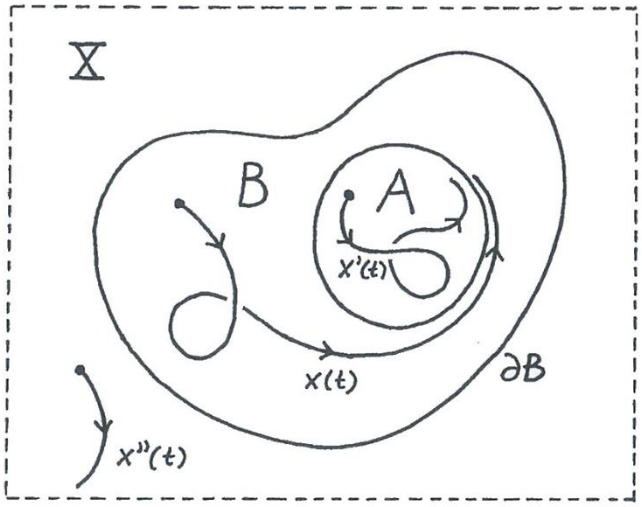
Comparing with the method of eigenvector decomposition gives:

$$\sum_{k=1}^{n} h_k = \sum_{k=1}^{n} \frac{\partial \dot{x}_k}{\partial x_k}$$

which reflects the theorem from linear algebra that the trace of matrix (sum of doagonal lelements) in this case the Jacobian is independent of the choice of basis. For the damped pendulum:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{l}\sin x_1 - \gamma x_2 \end{cases} \quad \rightarrow \quad \sum_{k=1}^{n} \frac{\partial \dot{x}_k}{\partial x_k} = \frac{\partial}{\partial x_1}(x_2) + \frac{\partial}{\partial x_2}\left(-\frac{g}{l}\sin x_1 - \gamma x_2\right) = -\gamma \quad \text{(as before)}$$

Attractors



If the volume contracts at an exponential rate we expect an ensemble of orbits to contract to a vanishing volume in n-space. Time to take a closer look at attractors that are not fixed points, cycles or quasi-periodic.

An **attractor** A has a **basin of attraction** B where trajectories $x(t)$ reach A asymptotically as $t \to \infty$. Trajectories that start in A, $x'(t)$ come close to all points of A and trajectories $x''(t)$ outside B never reach B (or A). ∂B is the **basin boundary**.

A closed set A in the set X of states $\boldsymbol{x}$ is an attractor if and only if

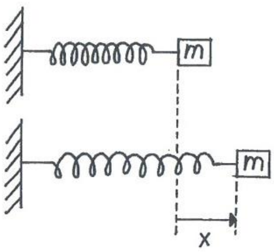
1. A is invariant, $\boldsymbol{x}(t_1) \in A \Rightarrow \boldsymbol{x}(t_2) \in A$ for all $t_2 > t_1$
2. There is an open set B such that $B \supset A$ and $\boldsymbol{x}(t) \in B \Rightarrow \lim_{t\to\infty} x(t) \in A$
3. A is minimal, no smaller set $A' \subset A$ fulfills condition 1 and 2.

The largest possible B in condition 2 is the basin of attraction.
The phase space volume B asymptotically shrinks to A.
$\mathrm{Vol}(A) < \mathrm{Vol}(B) \Rightarrow$ Only dissipative systems can have attractors.
Attractors with non-integer Hausdorff dimension (fractal structure) are called **strange attractors**.

Chaotic dynamics in dissipative systems is usually associated with strange attractors.

The Duffing oscillator

A basic example of nonlinearity, chaotic dynamics and a strange attractor is the **Duffing oscillator**, a mass attached to a spring with non-linear restoring force.



Newton's lax $m\ddot{x} = F_{\text{tot}} = F_{\text{Restore}} + F_{\text{Damping}} + F_{\text{External}}$
Assume F_R to be 'symmetric' under compression/extension'
$F_R(-x) = -F_R(x)$ and $\mathrm{sgn}(F_R(x)) = -\mathrm{sgn}(x)$ (opposite direction of extension)
Damping force: $F_D = -\gamma\dot{x}$
External driving force: $F_E = \alpha \sin \omega t$

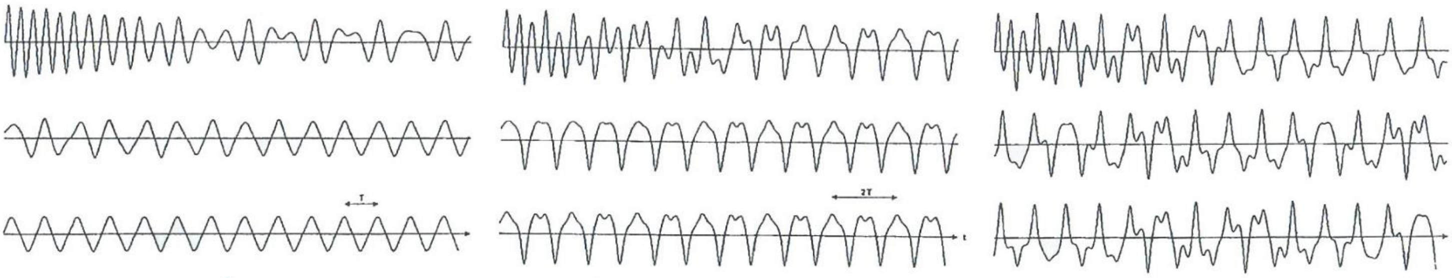
$$m\ddot{x} = F_R(x) - \gamma\dot{x} + \alpha \sin \omega t$$

A Duffing oscillator is a spring with simplest non-linear restoring force $F_R = -\beta x^3 = k(x) \cdot x$, with $k(x) = \beta x^2$, a spring that get harder as it stretches or compresses, a hardening spring, $m\ddot{x} + \gamma\dot{x} + \beta x^3 = \alpha \sin \omega t$. Duffing oscillators have been used to model skyscrapers and oil rigs. With appropriately chosen units it can be written

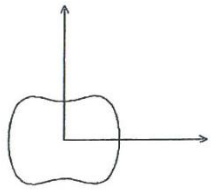
$$\ddot{x} + c\dot{x} + x^3 = A \sin t \quad \rightarrow \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1^3 - cx^2 + A \sin x_3 \\ \dot{x}_3 = 1 \end{cases}$$

Lie derivative: $\sum_k \partial_k \dot{x}_k = -c < 0$

As A varies there will be periodic attractors followed by period doublings and later a strange attractor.

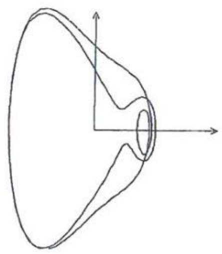


$A = 1$



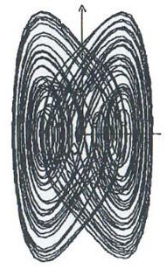
Time series $x(t)$ and 1-loop attractor in reduced phase space.

$A = 5$

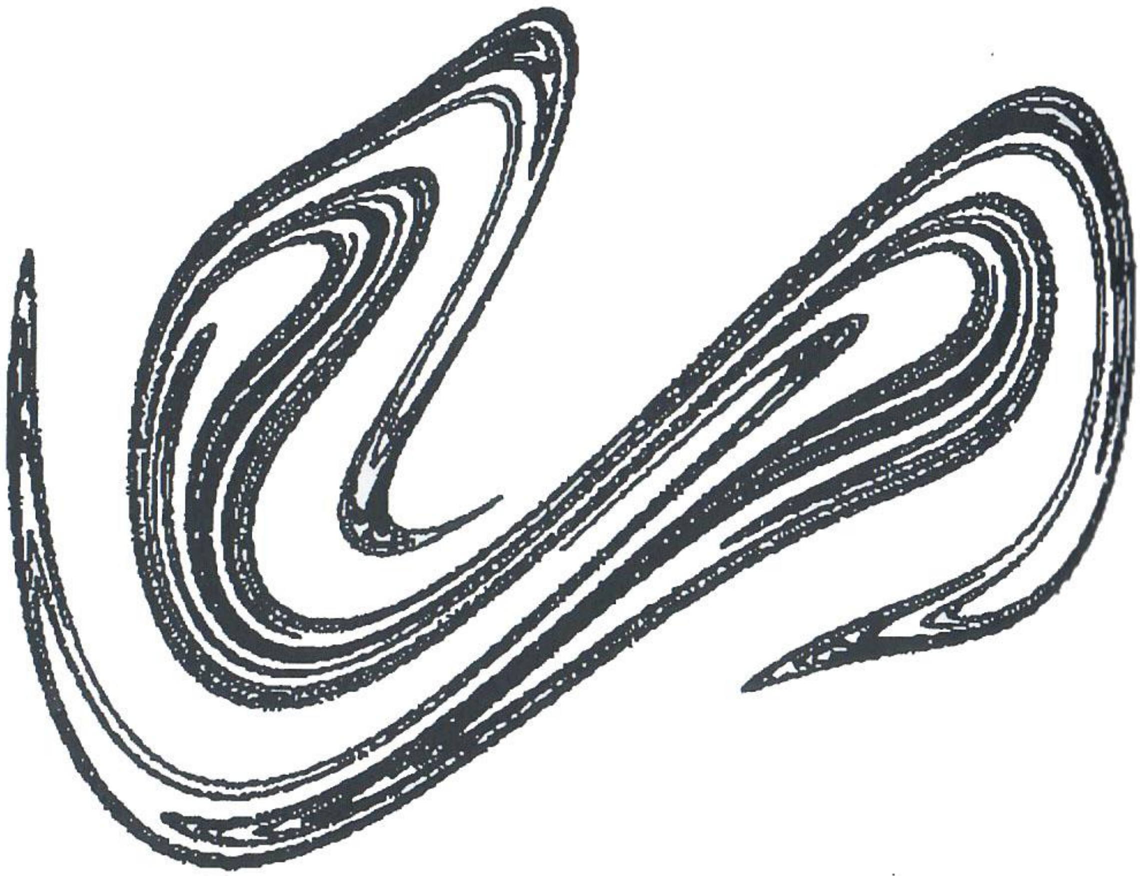


2 -loop attractor after bifurcation.

$A = 7.5$



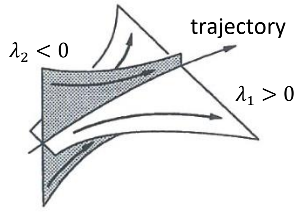
Non-periodic strange attractor.



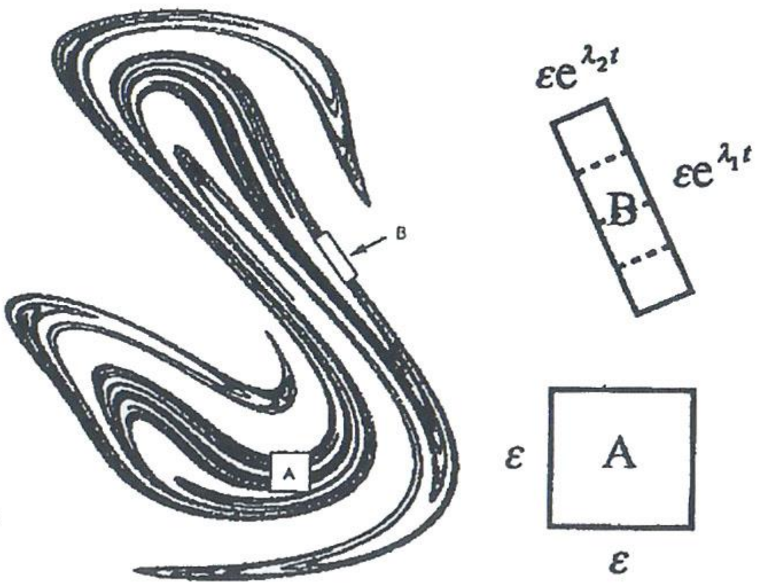
Poincaré section for the trajectory of the Duffing oscillator with $A = 7.5$

The Kaplan-Yorke conjecture

The box dimension D_B of a strange attractor (S) and the Lyapunov exponents are related. For a 3-dimensional phase space with a Poincare map in two dimensions with Lyapunov exponents λ_1 (positive) and λ_2 (negative) there is the **Kaplan-Yorke conjecture** which states that: $D_B(S) = 1 - \frac{\lambda_1}{\lambda_2}$



For 2D-maps $\left.\begin{matrix} |\delta A(N)| \approx |\delta A(0)| \cdot e^{(\lambda_1+\lambda_2)N} \\ \lambda_1 + \lambda_2 < 0 \end{matrix}\right\} \Rightarrow 1 < D_B < 2$



Cover S with squares of side ε, then the minimum number of squares required to cover S is $N(\varepsilon)$.

After N steps and time t for the Poincaré map the square has been stretched by a factor $e^{\lambda_1 t}$ and compressed in the other direction by a factor $e^{\lambda_2 t}$.

Cover S with boxes of size the short edge of B,

$$\tilde{\varepsilon} = \varepsilon e^{\lambda_2 t} \rightarrow t = \frac{1}{\lambda_2}\ln\frac{\tilde{\varepsilon}}{\varepsilon}$$

All boxes in the initial covering has been mapped all over S, the number of boxes of side $\tilde{\varepsilon}$ will be:

$$N(\tilde{\varepsilon}) = \frac{\varepsilon e^{\lambda_1 t}}{\varepsilon e^{\lambda_2 t}} N(\varepsilon) = e^{(\lambda_1-\lambda_2)t} N(\varepsilon)$$

$$\frac{N(\tilde{\varepsilon})}{N(\varepsilon)} = e^{(\lambda_1-\lambda_2)t} = e^{(\lambda_1-\lambda_2)/\lambda_2 \cdot \ln(\tilde{\varepsilon}/\varepsilon)} = \left(\frac{\tilde{\varepsilon}}{\varepsilon}\right)^{-\left(1-\frac{\lambda_1}{\lambda_2}\right)}$$

$$D_B(S) = 1 - \frac{\lambda_1}{\lambda_2}$$

Numerical experiments support the conjecture.
For the Duffing oscillator with $A = 7.5$ there is agreement between clalculations based on log-log plots of the number of boxes needed to cover S and the Kaplan-Yorke conjecture, $D_B(S) = 1.67 - 1.68$.

Dimension of the strange attractor of the Lorenz system

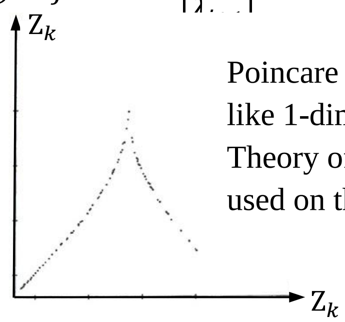
The Lie Derivative is:

$$\frac{\partial}{\partial X}(-aX + aY) + \frac{\partial}{\partial Y}(rX - Y - XZ) + \frac{\partial}{\partial Z}(XY - bZ) = -a - 1 - b = -41/3 \ \ (a = 10, b = 8/3, r = 28)$$

This gives a contraction factor of 10^{-6} at $\Delta t = 1$.
The largest Lyapunov exponent is $\lambda_1 = 0.96$
There is no contraction along the trajectory so $\lambda_2 = 0$ (parametrized by t).
$\lambda_1 + \lambda_2 + \lambda_3$ equals the Lie derivative, if it's constant this gives $\lambda_3 \approx -14.63$

The generalized Kaplan-Yorke for higher dimensions is: $D_B(S) = j + \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_j}{|\lambda_{j+1}|}$

with Lyapunov exponents in falling order and
j the index for smallest non-negative exponent.
Lorenz system: $j = 2 \quad \rightarrow \quad D_B(S) = 2 + \frac{0.96+0}{|-14.63|} = 2.07$
In good agreement with direct calculation which gives dimension 2.06 $\rightarrow D = 1.06$ for a Poincare section.



Poincare section for Z-plane is like 1-dimensional tent map. Theory of 1D-iterations can be used on the continuous case.